VOLUME 79 SEPARATE No. 331

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

NOVEMBER, 1953

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

DEFLECTION OF LAMINATED BEAMS

by L. G. Clark

Presented at
New York City Convention
October 19-22, 1953

ENGINEERING MECHANICS DIVISION

(Disscusion open until March 1, 1954)

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

THIS PAPER

--represents an effort of the Society to deliver technical data direct from the author to the reader with the greatest possible speed.

Readers are invited to submit discussion applying to current papers. For this paper the final closing dead line appears on the front cover.

Those who are planning papers or discussions for "Proceedings" will expedite Division and Committee action measurably by first studying "Publication Procedure for Technical Papers" (Proceedings — Separate No. 290). For free copies of this Separate—describing style, content, and format—address the Manager, Technical Publications, ASCE.

This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
PAPERS

DEFLECTION OF LAMINATED BEAMS

By L. G. Clark[1]

[1] Instructor, Dept. of Eng. Mechanics, Univ. of Michigan, Ann Arbor, Mich.

Synopsis

A theory is developed to determine the deflection of a beam composed of two or more laminations and riveted or fixed at certain points along the beam. The laminations can be of different sizes and materials but must be long in comparison with the cross-sectional dimensions of the beam. Although the problem investigated is based on only two laminations, the same approach can be used to provide for any number of laminations. The final result is in a form such that particular problems can be solved in a manner similar to the area moment method. It is shown that for two laminations joined at an infinite number of points the problem reduces to that of a simple beam. Two different cases of simply supported beams were tested and the satisfactory results are presented.

Introduction

Notation.—The letter symbols adopted for use in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically for convenience of reference in the Appendix.

It is known that a beam composed of laminations placed freely upon one another deflects more under a given load than does a solid beam of the cross section under the same loading conditions. The difference in deflection is caused by the fact that in the solid beam all longitudinal planes can support shear, whereas a freely laminated beam cannot resist shearing load on the planes that separate the individual laminations. This inability to resist horizontal shear has led to the use of various devices such as riveting, welding, or bolting, to connect the laminations so that the laminated beam can be made to act similarly to a solid beam. The problem of building such a beam is not new to the engineer. Unfortunately, however, the general feeling seems to be that as long as the method of joining the laminations will resist the longitudinal shear, as predicted by ideal beam theory, the problem is solved. That is, a built-up beam seems to be a result of guesswork in the matter of deflections.

Two Laminations Joined at Arbitrary Points

The analysis to be developed is easily extended to the case of a beam composed of more than two laminations. However, such an extension involves extensive algebraic computations which will do little to aid in the understanding of the problem. The analysis will determine the slope and deflection relationships between two points of the beam. These points will be chosen where the laminations are joined and there will be no joining of the laminations between the chosen joints.

The following assumptions are made concerning the two laminations shown in Fig. 1(a):

1. The deflections are small.
2. Any separation of the laminations is small in comparison with the distances a_1 and a_2. The symbol a_1 denotes the distance from the neutral axis of the upper lamination to the contact plane between the laminations. The distance from the neutral axis of the lower lamination to the contact plane between the laminations is represented by a_2. The subscripts "1" and "2" will be used in this manner; that is, the subscript "1" will refer to the upper of two laminations and the subscript "2" will refer to the lower lamination. It is important to note that separation is not neglected but merely assumed to be small.
3. The total elongation of the contact plane is the same for both laminations. This does not imply that the strain is the same at corresponding points of the laminations, but only that the integrated effect is the same.
4. There is no friction between the contact surfaces. This, of course, is not true in a real problem, but the assumption is made because the friction cannot be analytically determined. More important is the fact that any forces caused by friction will serve to improve rather than harm the characteristics of the beam.

It can be seen from a consideration of the equilibrium of the free-body shown in Fig. 1(b) (the right half of the beam shown in Fig. 1(a)) that

$$F_1 = F_2 = F \qquad (1)$$

and the forces are constant over the length L. Letting

$$\bar{a} = a_1 + a_2 \qquad (2)$$

and taking moments with respect to the left end of the free-body,

$$M_1 + M_2 = V\,(L - x) - F\,\bar{a} + M + \int_0^{L-x} w\,(\eta)\,\eta\,d\eta \qquad (3)$$

in which the integral is the bending moment caused by the distributed load. The integral is a function of x and will be designated by

$$\int_0^{L-x} w\,(\eta)\,\eta\,d\eta = f(x) \qquad (4)$$

The following representations will also be used:

$$\int f(x)\, dx = g(x) \qquad (5a)$$

and

$$\int g(x)\, dx = h(x) \qquad (5b)$$

From Eqs. 4 and 5 it will be noted that

$$f(L) = g(L) = h(L) = 0 \qquad (6)$$

Eq. 3 can be expressed as

$$M_1 + M_2 = V\,(L - x) - F\,\bar{a} + M + f(x) \qquad (7)$$

Elongation of the Contact Plane.—From ordinary beam theory the strain, γ, of a longitudinal fiber at the contact plane of the upper lamination is

$$\gamma_1 = -\frac{M_1\, a_1}{E_1\, I_1} + \frac{F}{A_1\, E_1} \qquad (8a)$$

and for the lower lamination

$$\gamma_2 = \frac{M_2\, a_2}{E_2\, I_2} - \frac{F}{A_2\, E_2} \qquad (8b)$$

in which E is the modulus of elasticity, I denotes the moment of inertia of the cross section, A is the cross-sectional area, and tensile forces are considered to be positive.

For the total elongation of the two laminations at the contact plane to be equal,

$$\int_0^L \left(\frac{F}{A_1\, E_1} - \frac{M_1\, a_1}{I_1\, E_1}\right) dx = \int_0^L \left(\frac{M_2\, a_2}{E_2\, I_2} - \frac{F}{A_2\, E_2}\right) dx \qquad (9)$$

from which

$$L\, F \left(\frac{1}{A_1\, E_1} + \frac{1}{A_2\, E_2}\right) = \int_0^L \frac{M_2\, a_2}{E_2\, I_2}\, dx + \int_0^L \frac{M_1\, a_1}{E_1\, I_1}\, dx \qquad (10)$$

Because

$$\frac{M_1}{I_1\, E_1} = \frac{d^2 y_1}{dx^2} \qquad (11a)$$

and

$$\frac{M_2}{I_2\, E_2} = \frac{d^2 y_2}{dx^2} \qquad (11b)$$

then

$$a_1 \int_0^L \frac{d^2 y_1}{dx^2}\, dx = a_1 \left[\left(\frac{dy_1}{dx} \right)_{x=L} - \left(\frac{dy_1}{dx} \right)_{x=0} \right] \qquad (11c)$$

and

$$a_2 \int_0^L \frac{d^2 y_2}{dx^2}\, dx = a_2 \left[\left(\frac{dy_2}{dx} \right)_{x=L} - \left(\frac{dy_2}{dx} \right)_{x=0} \right] \qquad (11d)$$

in which y is the deflection of the beam from an unloaded position.

From Fig. 2 it can be seen that

$$\left(\frac{dy_1}{dx} \right)_{x=0} = \left(\frac{dy_2}{dx} \right)_{x=0} = \theta_0 \qquad (12a)$$

and

$$\left(\frac{dy_1}{dx} \right)_{x=L} = \left(\frac{dy_2}{dx} \right)_{x=L} = \theta_L \qquad (12b)$$

Substitution of Eqs. 12 into Eqs. 11c and 11d and subsequent substitution of Eqs. 11c and 11d into Eq. 10 results in

$$F\, L \left(\frac{1}{A_1\, E_1} + \frac{1}{A_2\, E_2} \right) = a_1\, (\theta_L - \theta_0) + a_2\, (\theta_L - \theta_0) \qquad (13a)$$

from which

$$F\, L \left(\frac{1}{A_1\, E_1} + \frac{1}{A_2\, E_2} \right) = (a_1 + a_2)\, (\theta_L - \theta_0) = \bar{a}\, (\theta_L - \theta_0) \qquad (13b)$$

From Fig. 2 it can also be seen that

$$\theta_L - \theta_0 = \phi \qquad (14a)$$

With

$$b = \frac{1}{A_1\, E_1} + \frac{1}{A_2\, E_2} \qquad (14b)$$

Eq. 13b becomes

$$F\, L\, b = \bar{a}\, \phi \qquad (15)$$

Eq. 7 can be expressed as

$$E_1\, I_1 \frac{d^2 y_1}{dx^2} + E_2\, I_2 \frac{d^2 y_2}{dx^2} = (V\, L - F\, \bar{a} + M) - V\, x + f(x) \qquad (16)$$

Integration of Eq. 16 results in

$$E_1\, I_1 \frac{dy_1}{dx} + E_2\, I_2 \frac{dy_2}{dx} = (V\, L - F\, \bar{a} + M)\, x - \frac{V\, x^2}{2} + g(x) + K_a \qquad (17)$$

in which K_a is the constant to integration. At x equal to zero, Eq. 12a is applicable and

$$K_a = \theta_0\, (E_1\, I_1 + E_2\, I_2) - g(0) \qquad (18)$$

If

$$c = (E_1\, I_1 + E_2\, I_2) \qquad (19)$$

Eq. 17 becomes

$$E_1 I_1 \frac{dy_1}{dx} + E_2 I_2 \frac{dy_2}{dx} = (V L - F \bar{a} + M) x - \frac{1}{2} V x^2 + g(x) - g(0) + c \theta_0 \quad (20)$$

Integrating Eq. 20 results in

$$E_1 I_1 y_1 + E_2 I_2 y_2 = \frac{1}{2} (V L - F \bar{a} + M) x^2 - \frac{1}{6} V x^3 + h(x) - g(0) x + c \theta_0 x + K_b \quad (21)$$

in which K_b is the integration constant. At x equal to zero,

$$y_1 = y_2 = y_0 \quad (22)$$

Thus, from Eq. 21,

$$K_b = c y_0 - h(0) \quad (23)$$

and Eq. 21 becomes

$$E_1 I_1 y_1 + E_2 I_2 y_2 = \frac{1}{2} (V L - F \bar{a} + M) x^2 - \frac{1}{6} V x^3 - g(0) x - h(0) + h(x) + c \theta_0 x + c y_0 \quad (24)$$

Although Eq. 24 is a deflection equation, it is of no practical use in this form. At x equal to L, the boundary conditions are

$$\left(\frac{dy_1}{dx}\right)_{x=L} = \left(\frac{dy_2}{dx}\right)_{x=L} = \theta_L \quad (25a)$$

and

$$(y_1)_{x=L} = (y_2)_{x=L} = y_L \quad (25b)$$

Thus Eqs. 20 and 24 become

$$c \theta_L = (V L - F \bar{a} + M) L - \frac{1}{2} V L^2 - g(0) - c \theta_0 \quad (26a)$$

and

$$c y_L = \frac{1}{2} (V L - F \bar{a} + M) L^2 - \frac{1}{6} V L^3 - g(0) L - h(0) + c \theta_0 L + c y_0 \quad (26b)$$

or

$$0 = (V L - F \bar{a} + M) L - \frac{1}{2} V L^2 - g(0) - c (\theta_L - \theta_0) \quad (27a)$$

$$0 = \frac{1}{2} (V L - F \bar{a} + M) L^2 - \frac{1}{6} V L^3 - g(0) L - h(0) - c (y_L - \theta_0 L - y_0) \quad (27b)$$

and from Eq. 15,

$$F = \frac{\bar{a}}{L b} \phi \quad (28)$$

From Fig. 2 it can be seen that

$$\delta = y_L - y_0 - \theta_0 L \quad (29)$$

Therefore, Eq. 27*b* reduces to

$$0 = (V L - F \bar{a} + M) L - \frac{1}{2} V L^2 - g(0) - c \phi \quad (30a)$$

and

$$0 = \frac{1}{2} (V L - F \bar{a} + M) L^2 - \frac{1}{6} V L^3 - g(0) L - h(0) - c \delta \quad (30b)$$

Thus, Eqs. 30 and 28 can be solved for ϕ and δ in terms of the known quantities. Using Eqs. 28 and 30*a* one obtains

$$\phi = \frac{b}{\bar{a}^2 + c\,b}\left[\tfrac{1}{2}\,V\,L^2 + M\,L - g(0)\right] \qquad (31)$$

Eq. 31 is used to determine the angle between the tangents drawn at the ends of the section considered.

Using Eqs. 28 and 31 with Eq. 30*b* results in

$$\delta = \frac{V\,L^3}{c}\left[\frac{\bar{a}^2 + 4\,c\,b}{12\,(\bar{a}^2 + c\,b)}\right] + \frac{M\,L^2}{2}\left(\frac{b}{\bar{a}^2 + c\,b}\right) - \frac{g(0)}{c}\left[\frac{L\,(\bar{a}^2 + 2\,c\,b)}{2\,(\bar{a}^2 + c\,b)}\right] - \frac{h(0)}{c} \qquad (32)$$

Eq. 32 is used to determine the distance between the right jointing point (Fig. 2) and the tangent at the left end. Thus, Eqs. 31 and 32 can be used to find the deflections of any doubly laminated beam. The values of δ and ϕ are then used in the same manner as in the area moment method.

Unfortunately, the method presented herein will not give a solution for the deflection of the beam at points other than those that join the laminations. This is not a severe handicap since a close approximation can be made. The deflection y_1 is nearly equal to the deflection y_2, and by setting them equal, Eqs. 24, 31, and 32 will result in the solution for the deflection at any point.

Example No. 1.—To illustrate the simplicity of this procedure the problem of a cantilever beam composed of two laminations will be used. For ease of computation the laminations will be assumed to be of equal size and of the same material. These assumptions eliminate the need for subscripts to differentiate between the lamintations. The beam is shown in Fig. 3. From basic considerations.

$$I = \frac{1}{12}\,A\,h^2 \qquad (33a)$$

$$\bar{a} = h \qquad (33b)$$

$$b = \frac{2}{A\,E} = \frac{1}{6}\,\frac{h^2}{E\,I} \qquad (33c)$$

$$c = 2\,E\,I \qquad (33d)$$

and

$$b\,c = \tfrac{1}{3}\,h^2 \qquad (33e)$$

Because there are no distributed loads, Eqs. 4 and 5 equal zero and Eqs. 31 and 32 become

$$\phi = \frac{b}{\bar{a}^2 + b\,c}\left(\tfrac{1}{2}\,V\,L^2 + M\,L\right) \qquad (34a)$$

$$\delta = \frac{V\,L^3}{c}\left[\frac{\bar{a}^2 + 4\,b\,c}{12\,(\bar{a}^2 + c\,b)}\right] + \frac{M\,L^2}{2\,c}\left(\frac{c\,b}{\bar{a}^2 + c\,b}\right) \qquad (34b)$$

or (using Eqs. 33),

$$\phi = \frac{L}{8\,E\,I}\left(\tfrac{1}{2}\,V\,L + M\right) \qquad (35a)$$

$$\delta = \frac{7\,V\,L^3}{96\,E\,I} + \frac{M\,L^2}{16\,E\,I} = \frac{L^2}{16\,E\,I}\left(\frac{7}{6}\,V\,L + M\right) \qquad (35b)$$

From Fig. 3, it can be seen that

$$V_M = P \qquad (36a)$$

$$M_M = P\,L \qquad (36b)$$

$$V_N = P \qquad (36c)$$

and

$$M_N = 0 \qquad (36d)$$

Thus,

$$\theta_M = \frac{L}{8\,E\,I}\left(\frac{3}{2}\,P\,L\right) = \frac{3\,P\,L^2}{16\,E\,I} \qquad (37a)$$

$$\delta_M = y_M = \frac{L^2}{16\,E\,I}\left(\frac{7\,P\,L}{6} + P\,L\right) = \frac{13\,P\,L^3}{96\,E\,I} \qquad (37b)$$

and

$$\delta_N = \frac{L^2}{16\,E\,I}\left(\frac{7}{6}\,P\,L\right) = \frac{7\,P\,L^3}{96\,E\,I} \qquad (37c)$$

but

$$\Delta = y_M + \theta_M\,L + \delta_N \qquad (38a)$$

Therefore,

$$\Delta = \frac{13\,P\,L^3}{96\,E\,I} + \frac{3\,P\,L^3}{16\,E\,I} + \frac{7\,P\,L^3}{96\,E\,I} \qquad (38b)$$

or

$$\Delta = \frac{19\,P\,L^3}{48\,E\,I} \qquad (38c)$$

The deflection of a solid beam of the same dimensions and under the same load has the value,

$$\Delta_S = \frac{P\,(2\,L)^3}{3\,E\,I_s} \qquad (39a)$$

in which the subscript "s" denotes the solid beam, I_s equals $8\,I$ and is the moment of inertia of the entire cross section, and $2\,L$ is the total length of the solid beam. Therefore,

$$\Delta_s = \frac{8\,P\,L^3}{24\,E\,I} \qquad (39b)$$

It is of interest to note that the relative deflection between the solid and laminated beams is

$$\frac{\Delta}{\Delta_s} = 1.189 \qquad (40)$$

which means that for the laminated beam the deflection is only 19% more than the deflection of the solid beam.

If the two laminations had been riveted only at the end

$$\frac{\Delta}{\Delta_s} = 1.75 \qquad (41)$$

which indicates a substantial increase in stiffness as a result of the rivet at the center.

Example No. 2.—As an example, the case of the cantilever beam made of two laminations riveted only at the free end and supporting a uniformly distributed load will be investigated. This beam is shown in Fig. 4.

It can be seen that

$$g(0) = -\frac{w\,L^3}{6} \qquad (42a)$$

and

$$h(0) = \frac{w\,L^4}{24} \qquad (42b)$$

from which Eqs. 31 and 32 can be written as

$$\phi = \frac{b}{\bar{a}^2 + b\,c}\left(\tfrac{1}{2}\,V\,L^2 + M\,L + \frac{w\,L^3}{6}\right) \qquad (43a)$$

and

$$\delta = \frac{V\,L^3}{c}\left[\frac{\bar{a}^2 + 4\,c\,b}{12\,(\bar{a}^2 + c\,b)}\right] + \frac{M\,L^2}{2\,c}\left(\frac{c\,b}{\bar{a}^2 + c\,b}\right) + \left[\frac{L\,(\bar{a}^2 + 2\,c\,b)}{2\,(\bar{a}^2 + c\,b)}\right]\left(\frac{w\,L^3}{6\,c}\right) - \frac{w\,L^4}{24\,c} \qquad (43b)$$

Using the same notation as in Example No. 1 and realizing that

$$V = 0 \qquad (44a)$$

and

$$M = 0 \qquad (44b)$$

at the free end of the beam, then

$$\Delta = \frac{w\,L^3}{12\,E\,I}\left(\frac{L \times \frac{5}{3}\,h^2}{2 \times \frac{4}{3}\,h^2}\right) - \frac{w\,L^4}{48\,E\,I} \qquad (45a)$$

or

$$\Delta = \frac{3\,w\,L^4}{96\,E\,I} \qquad (45b)$$

For a solid beam,

$$\Delta_s = \frac{w\,L^4}{64\,E\,I} \qquad (46)$$

and

$$\frac{\Delta}{\Delta_s} = 2.00 \qquad (47a)$$

For the case in which a rivet is used in the beam midway between the ends and at the end,

$$\frac{\Delta}{\Delta_s} = 1.25 \qquad (47b)$$

Experimental Results

As a verification of the theory, two cases of a simply supported beam with a centrally placed concentrated load were solved and compared with experimental results. The results of Case I are shown in Fig. 5(*a*). Case I had steel laminations (16 in. by $1\frac{3}{4}$ in. by $\frac{5}{64}$ in.) joined by spot welds at the center of the span and at the two supports. The results of Case II are shown in Fig. 5(*b*). Case II had steel laminations (32 in. by $1\frac{1}{4}$ in. by $\frac{1}{8}$ in.) joined by tapered pins at the quarter points of the beam. In Figs. 5(*a*) and 5(*b*) curves have been plotted showing the theoretical deflections for a solid beam having the same dimensions and composed of the same material (steel, $E = 30 \times 10^6$ lb per sq in.) as the laminated beams. In Fig. 5(*b*) the measured-deflection points have had a curve drawn through them to show the divergence from the theoretical deflections. In Fig. 5(*a*) the points of measured deflection agreed closely with the theoretical deflection curve.

It is believed that, if larger loads had been used in Case I, the theoretical and actual curves would have separated at the higher loads as in Case II. At the time of the testing for Case I it was not considered necessary to go beyond the loads indicated. Graphite was placed between the laminations of the beam in Case II which would tend to eliminate any friction effect. It is believed that the deflections were larger than predicted because of the straining of the pins. Also, in Case I the laminations were spot welded, which would eliminate any stress concentration which would occur with tapered pins.

Limiting the Deflection

The question now arises as to the determination of the number of rivets (or joints) necessary to limit the deflection so that it will be within a certain percentage of the deflection of the solid beam of the same dimensions. This question will be investigated by the consideration of the cantilveer beam shown in Fig. 6. The beam consists of two laminations joined at uniformly distributed points l units apart. If there are $(m + 1)$ spaces between joints in the beam, the joints are numbered from right to left; and the right end of the beam is the zero end.

Then,

$$\Delta = \sum_{n=1}^{m} (\delta_n + n\, l\, \phi_n) + \delta_0 \qquad (48)$$

in which ϕ_n is the angle between the tangents at joint n, and joint $(n + 1)$ and δ_n denote the vertical distance between joint n and the tangent at joint $(n + 1)$. There are also the relations:

$$\frac{L_c}{m + 1} = l \qquad (49a)$$

$$V_n = P \qquad (49b)$$

and

$$M_n = P_n\, l \qquad (49c)$$

in which the subscript n denotes the values at the nth joint of the beam. From Eqs. 31 and 32,

$$\phi_n = \alpha\,(\tfrac{1}{2}\, V_n\, l^2 + M_n\, l) \qquad (50a)$$

in which

$$\alpha = \frac{b}{\bar{a}^2 + b\, c} \qquad (50b)$$

and

$$\delta_n = \beta\, V_n\, l^3 \frac{\alpha}{2}\, M_n\, l^2 \qquad (50c)$$

in which

$$\beta = \frac{\bar{a}^2 + 4\, c\, b}{12\, c\, (\bar{a}^2 + c\, b)} \qquad (50d)$$

Therefore,

$$\Delta = \sum_{n=1}^{m} [\beta\, V_n\, l^3 + \tfrac{1}{2}\,\alpha\, M_n\, l^2 + n\, l\,\alpha\,(\tfrac{1}{2}\, V_n\, l^3 + M_n\, l)] + \beta\, V_0\, l^3 \qquad (51a)$$

or

$$\Delta = \sum_{n=1}^{m} [V_n\, l^3\,(\beta + \tfrac{1}{2}\,\alpha\, n) + M_n\, l^2\,(\tfrac{1}{2}\,\alpha + n\,\alpha)] + \beta\, V_0\, l^3 \qquad (51b)$$

By use of Eqs. 49, Eq. 51*b* becomes

$$\Delta = P\, l^3 \sum_{n=1}^{m} [\beta + \tfrac{1}{2}\,\alpha\, n + \alpha\, n\,(\tfrac{1}{2} + n)] + \beta\, P\, l^3 \qquad (52a)$$

which is simplified to yield

$$\Delta = P \left(\frac{L_c}{m+1}\right)^3 \left[\beta + \sum_{n=1}^{m} (\beta + \alpha\, n + \alpha\, n^2)\right] \qquad (52b)$$

The ratio of deflections between the laminated beam and the solid beam, therefore, is

$$\frac{\Delta}{\Delta_s} = \frac{3\, E\, I_s}{(m+1)^3} \left[\beta\,(m+1) + \alpha \sum_{n=1}^{m} n\,(1+n)\right] \qquad (53a)$$

in which

$$\Delta_s = \frac{P\, L_c^3}{3\, E\, I_s} \qquad (53b)$$

Example No. 3.—For a beam composed of equal laminations,

$$\alpha = \frac{1}{8\, E\, I} \qquad (54a)$$

$$\beta = \frac{7}{96\, E\, I} \qquad (54b)$$

$$I_s = 8\, I \qquad (54c)$$

and

$$\frac{\Delta}{\Delta_s} = \frac{24\,E\,I}{(m+1)^3}\left[\frac{7\,(m+1)}{96\,E\,I} + \frac{1}{8\,E\,I}\sum_{n=1}^{m} n\,(1+n)\right] \quad \text{(54}d\text{)}$$

or

$$\frac{\Delta}{\Delta_s} = \frac{3}{(m+1)^3}\left[\frac{7\,(m+1)}{12} + \sum_{n=1}^{m} n\,(1+n)\right] \quad \text{(55)}$$

By use of Eq. 55 the following values have been computed:

m	$\frac{\Delta}{\Delta_s}$
0	1.750
1	1.188
2	1.083
3	1.047
4	1.030
5	1.021

The number of joining points can thus be chosen depending on the deflection desired.

It can be seen from the previously listed values for Δ/Δ_s that the advantage gained by adding an extra joining point decreases very rapidly as the number of joining points increases. Although this particular case is quite simple, it indicates a direct method of solving for the optimum number of rivets in a laminated beam. That is, by this theory there is no longer any need for a trial-and-error solution to find the necessary number of rivets to limit the deflection to a given amount. It should also be noted that friction will serve to reduce the values as given in the preceding tabulation. Consequently, the theory presents the worst situation possible.

More Than Two Laminations

The case of the laminated beams composed of more than two laminations will not be treated in detail. As might be suspected, the mathematical manipulation increases rapidly with the increase in number of laminations; therefore, only an indication of the procedure will be given.

Assuming that a beam is composed of k laminations, the section between two adjacent joining points will be investigated (Fig. 7). Taking as a free-body the section of length $(L - x)$, the following equations may be written:

$$F_1 + F_2 + F_3 + \cdots + F_k = 0 \quad \text{(56}a\text{)}$$

and

$$M_1 + M_2 + \cdots + M_k = V\,(L - x) + M + f(x) - [F_2\,c_2 + F_3\,a_3 + \cdots + F_k\,a_k] \quad \text{(56}b\text{)}$$

—in which

$$a_2 = d'_1 + d_2 \quad \text{(57}a\text{)}$$

$$a_3 = a_2 + d'_2 + d_3 \quad \text{(57}b\text{)}$$

and

$$a_i = a_{i-1} + d'_{i-1} + d_i \quad \text{(57}c\text{)}$$

and d_i is the distance between the neutral axis of the ith lamination and the contact surface of the lamination immediately below it.

Assuming the integral of the strains for the contact surface of two laminations to be equal, one can write:

$$\int_0^L \left(\frac{M_1 d'_1}{E_1 I_1} + \frac{F_1}{A_1 E_1}\right) dx = \int_0^L \left(-\frac{M_2 d_2}{E_2 I_2} + \frac{F_2}{A_2 E_2}\right) dx \quad \text{(58a)}$$

$$\int_0^L \left(\frac{M_2 d'_2}{E_2 I_2} + \frac{F_2}{A_2 E_2}\right) dx = \int_0^L \left(-\frac{M_3 d_3}{E_3 I_3} + \frac{F_3}{A_3 E_3}\right) dx \quad \text{(58b)}$$

and

$$\int_0^L \left(\frac{M_i d'_i}{E_i I_i} + \frac{F_i}{A_i E_i}\right) dx = \int_0^L \left(-\frac{M_{i+1} d_{i+1}}{E_{i+1} I_{i+1}} + \frac{F_{i+1}}{A_{i+1} E_{i+1}}\right) dx \quad \text{(58c)}$$

Thus, Eqs. 56, 57, and 58, the boundary conditions at $x = 0$,

$$y_1 = y_2 = y_3 = \cdots = y_k = y_0 \quad \text{(59a)}$$

$$\frac{dy_1}{dx} = \frac{dy_2}{dx} = \cdots = \frac{dy_k}{dx} = \theta_0 \quad \text{(59b)}$$

the boundary conditions at $x = L$,

$$y_1 = y_2 = \cdots = y_k = y_L \quad \text{(60a)}$$

$$\frac{dy_1}{dx} = \frac{dy_2}{dx} = \cdots = \frac{dy_k}{dx} = \theta_L \quad \text{(60b)}$$

and the same assumptions as for beams with two laminations will allow one to solve for ϕ and δ.

Proof of the Theory

The proof of the theory will be rigorous only to the extent that the area moment proposition is realized as the number of joined sections of a laminated beam approaches infinity. The proof will be offered for a doubly laminated beam—its extension to a more complicated system will be obvious.

If any two points on a doubly laminated beam with m joined points between them are considered, the case of a solid beam is approached by allowing m to increase indefinitely. Fig. 8(b) is an enlarged view of the section between joints n and $(n + 1)$ of Fig. 8(a) and Fig. 8(c) in the bending-moment diagram of the section. As l becomes smaller the loading can be considered as concentrated at the joined points.

Using Eqs. 31 and 32,

$$\phi = \sum_{n=0}^{m} \phi_n = \alpha \sum_{n=0}^{m} \left(\tfrac{1}{2} V_n l^2 + M_n l\right) \quad \text{(61a)}$$

and

$$\Delta = \sum_{n=0}^{m-1} (\delta_n + n\,\phi_n\, l) + \delta_0 \quad \text{(61b)}$$

Examination of the right side of Eq. 61a reveals that the term in parentheses is the area of the bending-moment diagram in Fig. 8(c). It is thus seen that ϕ is the sum of the bending-moment areas over the section, multiplied by α. This is the first area moment proposition except that the term α allows for a varying modulus of elasticity over the cross section.

By taking a term from Eq. 61b which corresponds to the section between joints n and $(n + 1)$—and with suitable manipulation—it can be shown that

$$\delta_n + n\,\phi_n\,l = (\beta\,V_n\,l^3 + \tfrac{1}{2}\,\alpha\,M_n\,l^2) + \alpha\,Z_n\,A_n \qquad (62a)$$

in which

$$Z_n = n\,l \qquad (62b)$$

and A_n is the area of the bending-moment diagram for the section.

If l approaches zero, and the higher order terms are canceled, Eq. 61b becomes

$$\delta = \sum_{n=0}^{m \to \infty} \alpha\,Z_n\,A_n \qquad (63a)$$

or (noting that $A_n = M\,dz$)

$$\delta = \int_0^L \alpha\,M\,Z\,dZ \qquad (63b)$$

This is the second area moment proposition except that α allows for a varying modulus of elasticity over the cross section, as before.

Conclusions

It should be noted that no attempt has been made to account for a shear deflection of the rivets or pins; this was not considered necessary for this analysis. Strains of this type could be brought into the problem by adding a term to Eqs. 9 and 58 which must express the shear strain in the rivets or pins that join the laminations.

APPENDIX

The following symbols, adopted for use in the paper and for the guidance of discussers, conform essentially with "American Standard Letter Symbols for Structural Analysis" (ASA Z10.8—1949), prepared by a Committee of the American Standards Association with Society representation, and approved by the Association in 1949:

A = cross-sectional area;
a = distance from neutral axis of the lamination to the contact plane;
$\bar{a}$ = the distance between neutral axes of adjacent laminations (defined by Eq. 2);
b = defined by Eq. 14b;
c = defined by Eq. 19;

E = modulus of elasticity;
F = axial load;
I = rectangular moment of inertia;
K_a = constant of integration in Eq. 17;
K_b = constant of integration in Eq. 20;
L = distance between joining points;
L_c = length of a cantilever beam;
l = uniform spacing of joints;
M = bending moment at a joint;
M_l = bending moment at the left end of Fig. 1(*a*);
P = concentrated load;
V = lateral shear at a joint;
V_l = lateral shear at the left end of Fig. 1(*a*);
w = uniform loading;
x = horizontal distance from arbitrary section;
y = deflection of the beam from the unloaded condition;
α = defined in Eq. 50*b*;
β = defined in Eq. 50*d*;
γ = strain of a longitudinal fiber;
Δ = total end deflection;
δ = vertical distance between tangent drawn through beam at $x = 0$ and beam at $x = L$;
η = distance coefficient for applied nonuniform loading;
θ = slope of tangent drawn to deflected beam; and
ϕ = defined by Eq. 14*a*.

The subscripts 1 and 2 are used to denote the upper and lower laminations, respectively. The subscripts 0 and L are used to signify the end of the beam at $x = 0$ and $x = L$, respectively. A solid beam of the same dimensions and material as the laminated beam is denoted by the subscript s.

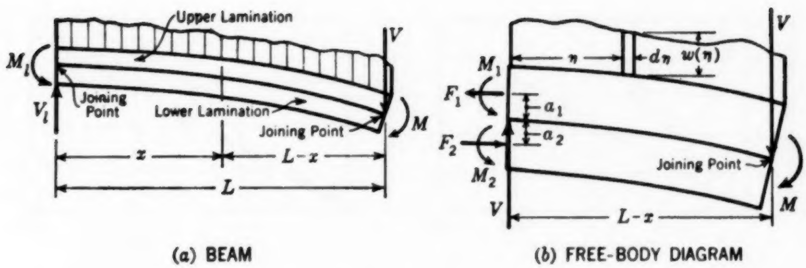


FIG. 1.—DOUBLY LAMINATED BEAM

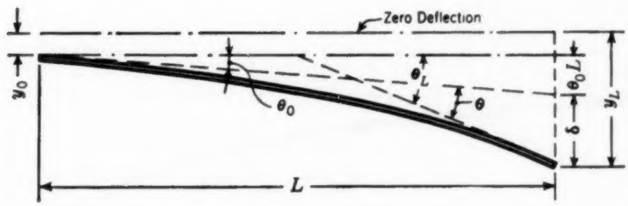


FIG. 2.—DEFLECTION DIAGRAM

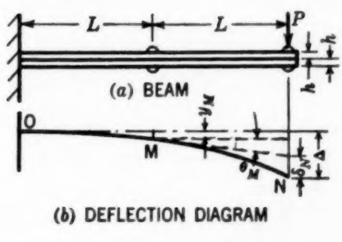


FIG. 3.—DOUBLY LAMINATED CANTILEVER BEAM SUBJECTED TO CONCENTRATED LOAD

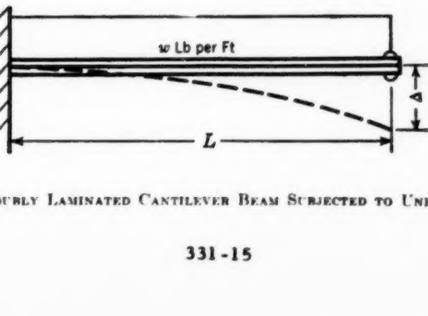


FIG. 4.—DOUBLY LAMINATED CANTILEVER BEAM SUBJECTED TO UNIFORM LOAD

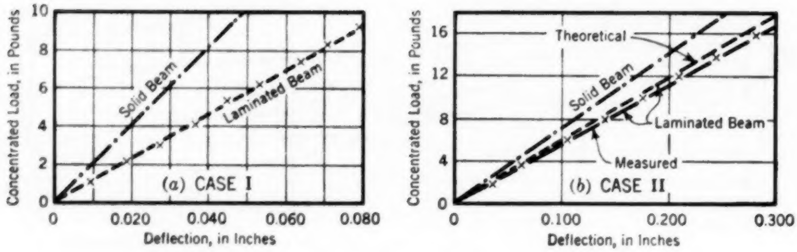


FIG. 5.—COMPARISON OF THEORETICAL AND EXPERIMENTAL DEFLECTIONS

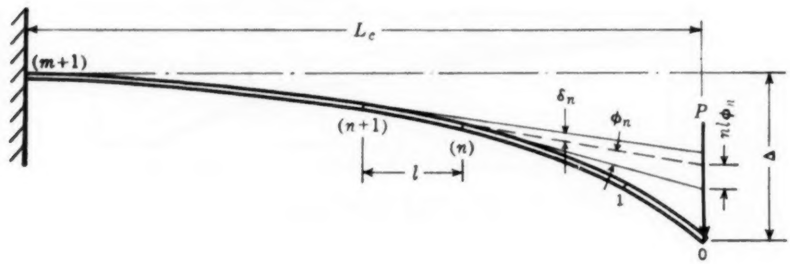


FIG. 6.—DOUBLY LAMINATED CANTILEVER BEAM JOINED AT MANY POINTS

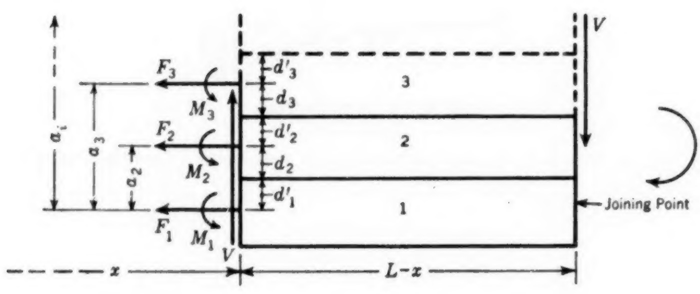
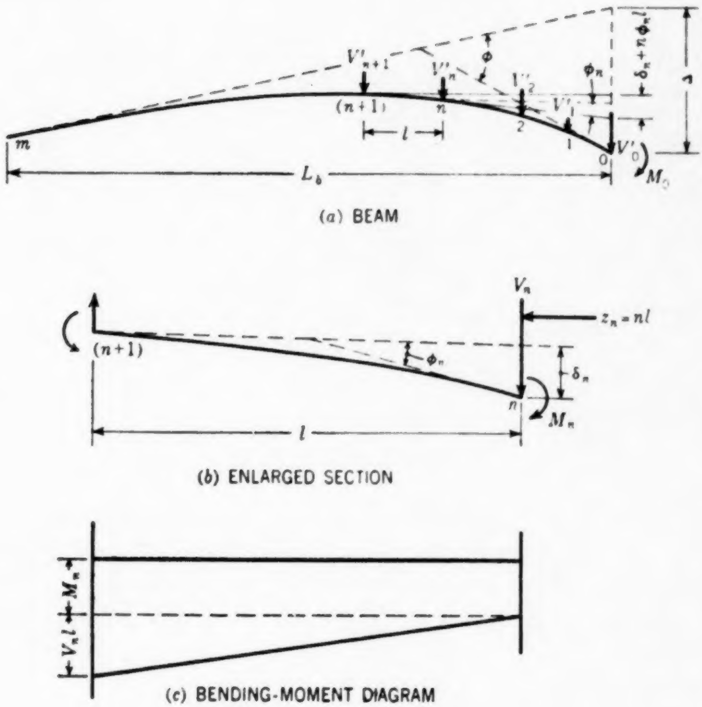


FIG. 7.—FREE-BODY DIAGRAM OF MULTIPLY LAMINATED BEAM



(a) BEAM

V_n
$z_n = nl$
$(n+1)$
ϕ_n
δ_n
n
M_n
l

(b) ENLARGED SECTION

M_n
$V_n l$

(c) BENDING-MOMENT DIAGRAM

FIG. 8 —DOUBLY LAMINATED BEAM